Announcements

1) Colloquium Tomorrow:
3:30 - 4:30 CB 2048
Mark Iwen, Michigan State
Applied Math - Fourier series

2) Guest speaker Thursday:
Joan Remski

3) 5b) on HW #4 is now extra credit. Hints to come. There will be a new 5b).

Definition: (Invertibility)

Let V be a vector space over $\mathbb{F}$, $T: V \to V$ be linear ($T \in \mathcal{L}(V)$).

T is invertible if

$\exists\ S: V \to V$ linear

with $\boxed{T \circ S = S \circ T = I_V}$

Observe: If $T \circ S = I_V$,

then T is surjective.

If $S \circ T = I_V$, then

T is injective.

Recall: If V is finite-dimensional over $\mathbb{F}$, then $T \in \mathcal{L}(V)$ is injective if and only if T is surjective.

Observation:

Let $V = \ell_2(\mathbb{N})$ over $\mathbb{C}$. An element of V is given by a sequence $(\alpha_i)_{i=1}^{\infty}$ of complex numbers with

$$\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty.$$

We can define

$T: V \to V$ by

$$T\left((\alpha_i)_{i=1}^{\infty}\right) = (\beta_i)_{i=1}^{\infty}$$

where $\beta_1 = 0$ and

$\beta_i = \alpha_{i-1} \ \forall \ i \geq 2.$

This "shifts" $(\alpha_i)_{i=1}^{\infty}$

one unit to the right.

T is injective

since if $T((\alpha_i)_{i=1}^{\infty})$

$= (0, 0, 0, \cdots)$,

then $\beta_i = \alpha_{i-1} = 0 \quad \forall i \geq 2$

$\Rightarrow \alpha_i = 0 \quad \forall i \geq 1$.

However, T is not surjective, since

$$\mathrm{Ran}(T) = \{ (\beta_i)_{i=1}^{\infty} \mid \beta_1 = 0 \}.$$

So in infinite dimensions, $\exists$ linear maps that are injective but not surjective (and vice-versa).

Temporary Solution

Only consider linear maps between finite-dimensional spaces.

Notation for Matrix Form

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. If $T \in \mathcal{L}(V, W)$, we can express T as a matrix, but the matrix is dependent on the choice of basis!

Let β be a basis for V and let γ be a basis for W. The notation for the matrix of T with respect to β and γ is

$$\boxed{[T]_\beta^\gamma}$$

Example 1: (2×2)

Let $S, T \in \mathcal{L}(\mathbb{R}^2)$,

$$S((x,y)) = (x-4y,\ y+8x)$$

$$T((x,y)) = (5x+2y,\ x+3y)$$

$$(S \circ T)((x,y)) = S((5x+2y,\ x+3y))$$

$$= ((5x+2y)-4(x+3y),\ x+3y+8(5x+2y))$$

$$= (x-10y,\ 41x+19y)$$

With respect to the standard basis, the matrix of T is $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$

and that of S is

$$\begin{bmatrix} 1 & -4 \\ 8 & 1 \end{bmatrix}.$$

The matrix of

$S \circ T$ is

$$\begin{bmatrix} 1 & -10 \\ 41 & 19 \end{bmatrix}.$$

Note that

$$\underbrace{(1,-4)}_{\text{first row of } S} \cdot \underbrace{(5,1)}_{\text{first column of } T} = 1$$

This is the (1,1) entry of $S \circ T$'s matrix.

Similarly,

$$(1,-4) \cdot (2,3) = -10$$

(1,2) entry of $S \circ T$

$$(8,1) \cdot (2,3) = 19$$

(2,2) entry of $S \circ T$

$$(8,1) \cdot (5,1) = 41$$

(2,1) entry of $S \circ T$

This shows that for our S and T, there is a "multiplication" on matrices such that if $\mathcal{E}$ = standard basis,

$$\boxed{[S \circ T]_{\mathcal{E}}^{\mathcal{E}} = [S]_{\mathcal{E}}^{\mathcal{E}} \cdot [T]_{\mathcal{E}}^{\mathcal{E}}}$$

Definition: (matrix product)

Let

$$A = (a_{i,j})_{i=1}^{n}{}_{j=1}^{m} \in M_{n\times m}(\mathbb{F})$$

and

$$B = (b_{t,s})_{t=1}^{m}{}_{s=1}^{r} \in M_{m\times r}(\mathbb{F})$$

We define the matrix product $A \cdot B$ as the matrix in $M_{n \times r}(\mathbb{F})$ with

$$\boxed{(A \cdot B)_{i,s} = \sum_{t=1}^{m} a_{i,t} b_{t,s}}$$

$$(A \cdot B)_{i,s}$$

$=$ the dot product of the i^{th} row of A with the s^{th} column of B.

Proposition: Let V, W, U be finite-dimensional vector spaces and let $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(V,W)$, $R \in (W,U)$, $\alpha \in \mathbb{F}$.

Let β be a basis for V, γ a basis for W, and δ a basis for U.

Then

a)

$$[T+S]_\beta^\gamma = [T]_\beta^\gamma + [S]_\beta^\gamma$$

b) $$[\alpha T]_\beta^\gamma = \alpha [T]_\beta^\gamma$$

c)

$$[R \circ T]_\beta^\delta = [R]_\gamma^\delta [T]_\beta^\gamma$$

Proof: exceptionally tedious.

Let's do a)

How do we get these matrices? For T, for example, if

$\beta = \{x_1, \ldots, x_n\}$

$\gamma = \{y_1, \ldots, y_n\}$

If $1 \le i \le n$,

$$T(x_i) = \sum_{j=1}^{m} \alpha_{j,i} y_j$$

for $\alpha_{j,i} \in \mathbb{F}$, $1 \le j \le m$.

We let the sequence

$(\alpha_{j,i})_{j=1}^{m}$ be the i^{th} column of $[T]_{\beta}^{\gamma}$.

Since

$$(T+S)(x_i)$$

$$= Tx_i + Sx_i$$

$$= \sum_{j=1}^{m} \alpha_{j,i}\, y_j + \sum_{j=1}^{m} \beta_{j,i}\, y_j$$

$$= \sum_{j=1}^{m} (\alpha_{j,i} + \beta_{j,i})\, y_j$$

and the coefficients

$(\alpha_{j,i})_{i=1}^{n}{}_{j=1}^{m}$ and

$(\beta_{j,i})_{i=1}^{n}{}_{j=1}^{m}$ are

uniquely determined,

we see

$$[T+S]_\beta^\gamma = [T]_\beta^\gamma + [S]_\beta^\gamma$$

no proof for the others! $\square$